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A mimetic finite difference discretization for the incompressible Navier–Stokes equations

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SUMMARY

The results of a mimetic finite difference discretization of the three-dimensional, incompressible Navier–Stokes equations are compared with more traditional finite difference schemes. The proposed method handles both momentum advection and diffusion in a vorticity-preserving manner and allows for simple treatment of rigid wall boundary conditions. The results obtained in various tests demonstrate the advantages of the proposed method. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Numerical methods for fluid flow preserving discrete analogs of some invariants of the equations of motion (such as mass, momentum, energy, enstrophy) were studied intensively for applications to large-scale atmospheric flows, see the finite difference schemes proposed, e.g. in [1, 2]. In more recent work, several of these properties have been extended to triangular meshes under mild regularity assumptions, see e.g. [3–5]. The development of numerical methods with discrete conservation properties can take advantage of the so-called *mimetic* finite difference schemes, for which discrete analogs of continuous identities hold, such as $\nabla\times\nabla\phi=0$, integration by parts formulae and the Helmholtz decomposition theorem. Examples of mimetic finite differences are given, e.g. by Hyman and Shashkov [6] and Nicolaides [7], in two- and three-dimensional frameworks, respectively.

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Preservation of discrete invariants is usually motivated by the desire of reproducing correct energy or enstrophy spectra, [3–5]. In this paper, we will take the viewpoint of [8] as a working hypothesis and we will investigate numerically vorticity-preserving discretizations for incompressible flow problems at the laboratory scale, in order to assess their potential advantages with respect to more common discretization approaches. Specifically, a marker and cell (MAC) type, mass and vorticity-preserving finite difference discretization of the three-dimensional Navier–Stokes equations is introduced, based on the concepts first proposed in [2] for the shallow water equations. A similar three-dimensional extension was first introduced in [9] for models of nonhydrostatic atmospheric flows. Vorticity preservation means that a consistent discrete vorticity equation can be achieved by application of a mimetic *curl* operator to the discrete momentum equation. As a consequence, the spatial semi-discretization preserves irrotational discrete initial data in the absence of viscosity. Furthermore, both the viscous term and rigid wall boundary conditions are discretized consistently in a vorticity-preserving manner. A number of numerical experiments show that the proposed discretization concept produces remarkable improvements with respect to conventional approaches, especially in regimes where localized vorticity production is taking place close to boundaries. This motivates further research and investigation, in order to achieve a more systematic assessment of the relative merits of the present approach with respect to energy-preserving methods such as those proposed in [4] and with respect to other finite volume and finite element discretizations.

2. A VORTICITY-PRESERVING SPATIAL DISCRETIZATION FOR THE NAVIER–STOKES EQUATIONS

The Navier–Stokes equations for a constant density, incompressible fluid can be formulated as

$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\omega}\times\mathbf{u} - \nabla(p+K) - \mu\nabla\times\boldsymbol{\omega} \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

where $\boldsymbol{\omega}=\nabla\times\mathbf{u}$ denotes vorticity and $K=\|\mathbf{u}\|^2/2$ denotes kinetic energy. Taking the curl of the momentum equation, an evolution equation for vorticity can also be obtained

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -\nabla\times[\boldsymbol{\omega}\times\mathbf{u}] + \mu\Delta\boldsymbol{\omega} \tag{3}$$

A discretization of Equations (1)–(2) is introduced, along the lines of popular discretization methods such as the MAC approach or the Arakawa C grid (see, e.g. [1]). Each cell is numbered at its center with indices i, j and k, for the x, y and z directions, respectively. The length of the cell sides in each directions is denoted by Δx_i, Δy_j and Δz_k and they are assumed to vary in their respective directions only. The cell volume is given by $V_{i,j,k}=\Delta x_i \Delta y_j \Delta z_k$ and staggered spacings $\Delta x_{i+1/2}$ are defined by arithmetic average. The discrete u velocity is defined at half integer i and integers j and k, v is defined at integers i, k and half integer j, while w is defined at integers i, j and half integers k. Finally, p and all other three-dimensional scalar variables are defined at integers i, j, k. At points where they are not defined, the discrete variables are generally computed by simple arithmetical mean of the nearest defined values. Averaged quantities will usually be denoted by an overbar. Finite difference operators are then introduced as customary in the MAC

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approach. The vorticity fluxes, instead, are naturally defined *via* the Stokes theorem at the faces of staggered control volumes, so that

$$\begin{aligned}
\omega^x_{i,j+1/2,k+1/2} &= \frac{w_{i,j+1,k+1/2}-w_{i,j,k+1/2}}{\Delta y_{j+1/2}} - \frac{v_{i,j+1/2,k+1}-v_{i,j+1/2,k}}{\Delta z_{k+1/2}} \\
\omega^y_{i+1/2,j,k+1/2} &= \frac{u_{i+1/2,j,k+1}-u_{i+1/2,j,k}}{\Delta z_{k+\frac{1}{2}}} - \frac{w_{i+1,j,k+1/2}-w_{i,j,k+1/2}}{\Delta x_{i+1/2}} \\
\omega^z_{i+1/2,j+1/2,k} &= \frac{v_{i+1,j+1/2,k}-v_{i,j+1/2,k}}{\Delta x_{i+1/2}} - \frac{u_{i+1/2,j+1,k}-u_{i+1/2,j,k}}{\Delta y_{j+1/2}}
\end{aligned} \tag{4}$$

A discrete *curl* operator can be defined for each cell as

$$\text{curl}(u,v,w)_{i,j,k} = (\omega^x_{i,j+1/2,k+1/2}, \omega^y_{i+1/2,j,k+1/2}, \omega^z_{i+1/2,j+1/2,k}) \tag{5}$$

These definitions are similar to those given, e.g. in [6] and have similar mimetic properties. A more complete description of the proposed numerical scheme can be found in the report [10]. Applying these definitions to the discretization of Equations (1)–(2) on a MAC-type grid yields the second-order accurate spatial discretization:

$$\begin{aligned}
\frac{\partial}{\partial t}u_{i+1/2,j,k} = &-\bar{\omega}^y_{i+1/2,j,k}\bar{w}_{i+1/2,j,k} + \bar{\omega}^z_{i+1/2,j,k}\bar{v}_{i+1/2,j,k} \\
&-\delta_x(p+\bar{K})_{i+1/2,j,k} \\
&+\mu[\delta_z(\omega^y)_{i+1/2,j,k} - \delta_y(\omega^z)_{i+1/2,j,k}]
\end{aligned} \tag{6}$$

$$\begin{aligned}
\frac{\partial}{\partial t}v_{i,j+1/2,k} = &-\bar{\omega}^z_{i,j+1/2,k}\bar{u}_{i,j+1/2,k} + \bar{\omega}^x_{i,j+1/2,k}\bar{w}_{i,j+1/2,k} \\
&-\delta_y(p+\bar{K})_{i,j+1/2,k} \\
&+\mu[\delta_x(\omega^z)_{i,j+1/2,k} - \delta_z(\omega^x)_{i,j+1/2,k}]
\end{aligned} \tag{7}$$

$$\begin{aligned}
\frac{\partial}{\partial t}w_{i,j,k+1/2} = &-\bar{\omega}^x_{i,j,k+1/2}\bar{v}_{i,j,k+1/2} + \bar{\omega}^y_{i,j,k+1/2}\bar{u}_{i,j,k+1/2} \\
&-\delta_z(p+\bar{K})_{i,j,k+1/2} \\
&+\mu[\delta_y(\omega^x)_{i,j,k+1/2} - \delta_x(\omega^y)_{i,j,k+1/2}]
\end{aligned} \tag{8}$$

$$\text{div}(u,v,w)_{i,j,k} = 0 \tag{9}$$

This approach allows to preserve mass and vorticity, but not kinetic energy. It extends to the three-dimensional, viscous, incompressible case, the techniques proposed in [3] for the discretization of the shallow water equations on a triangular geodesic grid. These were in turn inspired by the seminal paper [2]. In the two-dimensional inviscid case, the discretization coincides exactly with that of [2], if constant fluid thickness is assumed in the shallow water equations considered therein.

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3. NUMERICAL EXPERIMENTS

For the purpose of numerical tests discussed in this paper, a simple second-order Runge–Kutta time discretization was considered. The time discretization was performed along the lines of projection methods, with an explicit predictor step and a following pressure correction step, in which a Poisson equation is solved for pressure to ensure that the discrete divergence-free constraint is enforced. For all the numerical tests considered, relatively small values of the time step and of the Courant number were used, since the focus here is on the investigation of the properties of the spatial discretization.

Throughout this section, the results of the vorticity-preserving scheme, which will be referred to as Scheme 2, will be compared with those obtained in the same test cases with another finite difference method for the discretization of the nonlinear momentum equation, which we will denote as Scheme 1. More specifically, the centered finite difference method of [11] has been employed, coupled to the same time discretization described above. The spatial discretization of [11] is also mass conservative and uses the same MAC-type staggered grid and the same discretization of the divergence operator. It only differs from our approach in the approximation of the momentum equation, which does not preserve vorticity in the sense described above. The implementation of the finite difference method of [11] used for these tests had been validated previously in a number of laminar and turbulent flow simulations (see, e.g. [12]).

In the first numerical experiment, we have considered the benchmark test case proposed in [13], concerning the simulation of the flow around a square cylinder. We will focus on relatively low Reynolds numbers, for which a laminar flow regime is guaranteed. Strong vorticity production takes place at the obstacle corners, along with vortex shedding in the lee. Reference experimental results for this configuration are presented, e.g. in [14].

The flow has been simulated at Reynolds numbers $Re=250$ up to approximately 200 nondimensional time units. For these values of the Reynolds number the flow is still two dimensional, so that it is reasonable to employ a quite coarse resolution in the transversal y direction. The first

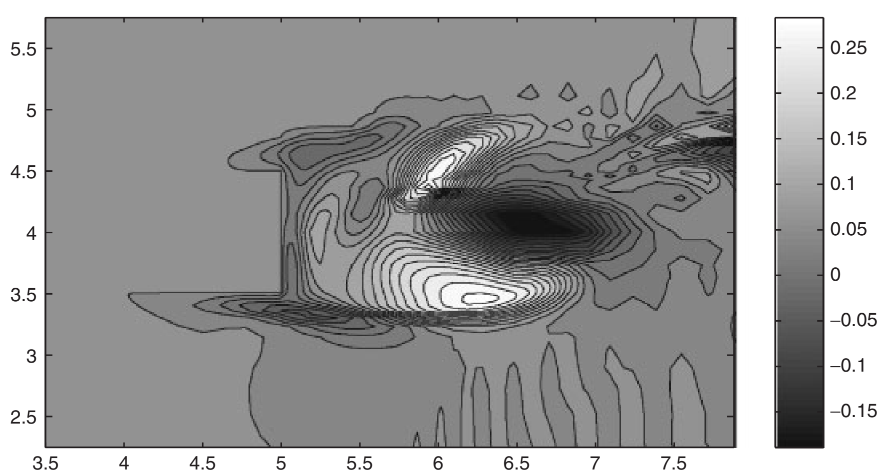

Figure 1. Transversal velocity field around the square cylinder at $Re=250$, Scheme 1.

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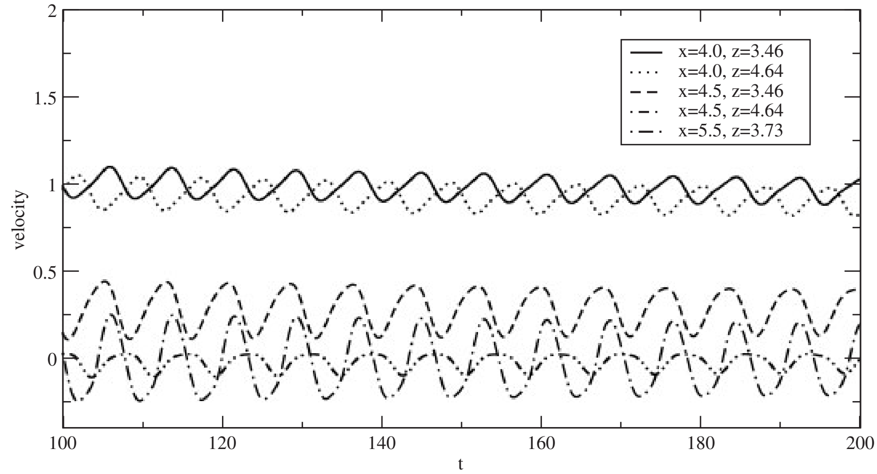

Figure 2. Longitudinal velocity component *versus* time for different positions around the square cylinder at $Re=250$, Scheme 2.

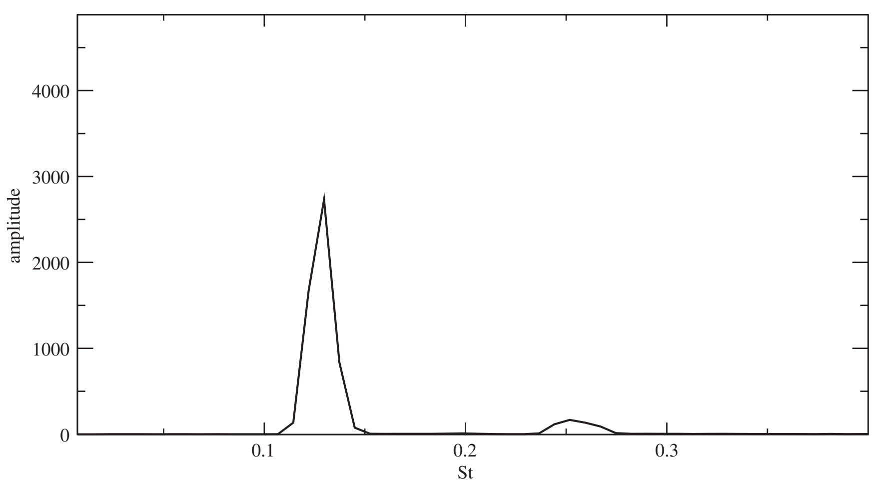

Figure 3. Strouhal number in the flow around the square cylinder at $Re=250$, Scheme 2.

striking difference between Schemes 1 and 2 is that application of the former produces spurious transversal velocities of the same order of magnitude of the inflow velocity. The spurious transversal v velocity components are shown in Figure 1. On the other hand, for Scheme 2, the transversal velocity component remains zero at machine accuracy. This seems to support the view that spurious vorticity production can be quite damaging for local accuracy when localized vorticity production occurs.

At $Re=250$, the frequency of the vortex shedding in the lee of the obstacle was also investigated for Scheme 2. This analysis could not be carried out for Scheme 1 since a statistical steady-state solution had not been reached yet at the end of the chosen simulation time. In Figure 2, the time

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evolution of the horizontal velocity component u is shown, as computed at various different points in the lee of the cylinder. All the time series display the same frequency, albeit with different phase shifts. The Strouhal number $St=\omega/UL$ has also been computed, see Figure 3, yielding a value of approximately $St=0.1297$ that is in good agreement with most of the values computed in [13] and reasonably close to the experimental values reported in [14].

4. CONCLUSIONS

These results seem to show that the method has considerable advantages with respect to more conventional approaches, especially in regimes where highly localized vorticity production is taking place close to boundaries, thus supporting the heuristic consideration put forward in [8]. As a consequence, it seems that there is a strong motivation for further research and analysis. In particular, we would like to carry out a more systematic assessment of the relative merits of the present approach with respect to energy-preserving methods such as those proposed in [4] and with respect to other finite volume and finite element discretizations.

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